

## REPORT DOCUMENTATION PAGE

Form Approved  
OMB No. 0704-0188

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing the collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing this burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22207-4302, and to the Office of Management and Budget, Paperwork Reduction Project (0704-0188), Washington, DC 20503.

1. AGENCY USE ONLY (Leave blank)	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	6 Jun 95	Final

4. TITLE AND SUBTITLE	5. FUNDING NUMBERS
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Final-Cost Estimates for Research & Development  
Programs Conditioned on Relized Costs

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8. PERFORMING ORGANIZATION  
REPORT NUMBER

## 9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Program Analysis and Evaluation  
Office of the Secretar of Defense  
1800 Defense Pentagon  
Washington DC 20301-1800

10. SPONSORING/MONITORING  
AGENCY REPORT NUMBER

## 11. SUPPLEMENTARY NOTES

## 12a. DISTRIBUTION/AVAILABILITY STATEMENT

Approved for Public release,  
Distribution is unlimited

## 12b. DISTRIBUTION CODE

## 13. ABSTRACT (Maximum 200 words)

We apply multiple model adaptive estimatin (MMAE), a proven method of system identification widely used in engineering applications, to the problem of determining Bayesian probability distributions of the final cost and completion time of on-going research and development (R&D) programs, conditioned on actual cost of work performed (ACWP) data. Modeling cumulative expenditures with Rayleigh distributions, we produce graphs of the results that give useful assessments of cost and schedule risks. The procedure is implemented in a convenient computer program. We give three examples of its application to actual data, and results of a Monte Carlo analysis verify the method.

Military Operations Reseach Society

MORS  
G3 Symposium

## 14. SUBJECT TERMS

Cost, R&D, Kalman Filter, MMAE

## 15. NUMBER OF PAGES

33

MORS Military Operations Reseach Society

## 16. PRICE CODE

17. SECURITY CLASSIFICATION  
OF REPORT

Unclassified

18. SECURITY CLASSIFICATION  
OF THIS PAGE

Unclassified

19. SECURITY CLASSIFICATION  
OF ABSTRACT

Unclassified

## 20. LIMITATION OF ABSTRACT

UL

NSN 7540-01-280-5500

Standard Form 298 (Rev. 2-89)  
Prescribed by ANSI Std. Z39-18  
298-102

DTIC QUALITY INSPECTED 1

19950804  
059

36K

# **Final-Cost Estimates for Research & Development Programs Conditioned on Realized Costs**

8/1/95

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## **Abstract**

We apply Multiple Model Adaptive Estimation (MMAE), a proven method of system identification widely used in engineering applications, to the problem of determining Bayesian probability distributions of the final cost and completion time of on-going Research and Development (R&D) programs, conditioned on actual cost of work performed (ACWP) data. Modeling cumulative expenditures with Rayleigh distributions, we produce graphs of the results that give useful assessments of cost and schedule risks. The procedure is implemented in a convenient spreadsheet. We give three examples of its application to actual data, and results of a Monte Carlo analysis verify the method.

## **1. Introduction**

Estimates of cost and duration for Research and Development (R&D) programs often increase significantly during the project. Development costs of the Concorde aircraft exceeded original estimates by more than a factor of five. In defense acquisition, where development

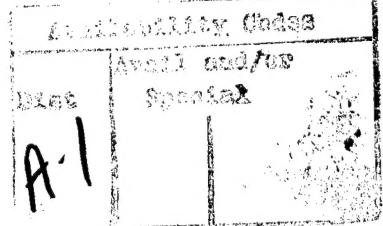
programs for major weapon systems—aircraft, tanks, missiles—often cost billions of dollars, some development programs' final costs and completion times have been twice the original projections.

R&D programs typically undergo periodic reviews, at which estimates of the cost-to-go are critical data for decisions on whether or not to continue. At such reviews, point estimates of final cost and completion times are not particularly helpful to management because of their uncertainty. Even a firm fixed-cost development contract does not guarantee a total final cost since requests for equitable adjustment often add substantially to the costs of a program.

At an intermediate review, management needs quantitative estimates of the cost and schedule *risks* of continuing the program. They need estimates of the probability distribution of final cost and completion times, conditioned on present knowledge, for example on expenditures to date. Knowing that available information indicates that final cost and completion times are likely to fall in relatively narrow intervals—or, conversely, that sets of costs and completion times occupying relatively broad intervals are all about equally likely—can greatly benefit decision making.

In this paper, we develop a method for determining Bayesian probability distributions of final cost and completion times of R&D programs, from data on incurred costs (specifically, from the actual cost of work performed (ACWP) data provided in cost performance reports). A spreadsheet that is convenient for use on microcomputers implements the algorithm. With this tool, management may easily access the cost and schedule risks inherent in continuing a R&D program.

The method that we apply, Multiple Model Adaptive Estimation (MMAE) [16,17], is widely used by scientists and engineers dealing with electronic and mechanical systems. MMAE



is a method for system identification, which is identifying the unknown properties of a system from observations to predict the system's future behavior. System identification is an extensively developed part of mathematical system theory. Since many tasks in cost analysis are system identification tasks, it seems helpful to apply that knowledge to them.

MMAE requires a model of the system studied, and in this paper we use the Rayleigh probability model for the time-history of expenditures in an R&D program. Several cost analysts studied the applicability of that model [1,2,6,7,10,11,12,20,21], and concluded that it represents R&D phases of major defense acquisition programs well.

MMAE involves the use of Kalman filters to estimate the state of a system, given noisy observations. A system's "state" is a set of parameters that describe its configuration fully, and determine its future evolution (given future inputs). For example, in Newtonian mechanics the state of a mass point is a set of three position coordinates and three velocity coordinates. In this paper, we define the state of a development project as its earned value, measured by ACWP.

The Kalman filter [13,14,15] uses a model of the system to project the Bayesian probability density of its state, conditioned on a set of noisy observations. The Kalman filter results are optimal for linear system models, Gaussian noises, and natural definitions of "optimal." The filter computations proceed iteratively and are computationally tractable.

Our application of MMAE determines the likelihood of various values of the two parameters of a Rayleigh model, based on the residuals from a set of Kalman filters. This allows us to produce graphs of the probability that the final cost or the completion time will not exceed any particular value. These graphs give managers a clear indication of the cost and schedule risk in continuing a development program.

We discuss the Rayleigh model and its applicability to R&D programs in Section 2. Section 3 presents the development of a dynamics model for earned value over time. We describe Kalman filters and MMAE as used in this application in Sections 4 and 5, respectively. In Section 6, we summarize the steps in the method. Section 7 contains sample applications, and a Monte Carlo analysis of the proposed technique is presented in Section 8. The paper concludes with a summary.

## **2. The Rayleigh Model**

Norden proposed that the Rayleigh distribution function can model expenditures for R&D programs [18]. He stated:

that there are regular patterns of manpower buildup and phase-out in complex projects. ... The cycles do not depend on the nature or work content of the project but seem to be a function of the way groups of engineers and scientists tackle complex technological development problems.

Norden derived the relationship based on the assumption that the effectiveness which problems are solved improves as a linear function of time. "Norden's description of the process is this: The rate of accomplishment is proportional to the pace of the work times the amount of work remaining to be done." [19] Putnam summarized testing of the Rayleigh model on estimating manpower for over 200 software development projects as follows:

Many of these also exhibit the same basic manpower pattern--a rise, peaking, and exponential tail off as a function of time. Not all systems follow this pattern. ... It is because manpower is applied and controlled by management. Management may choose to apply it in a manner that is suboptimal or contrary to system requirements. [20]

Within the Department of Defense, weapon system R&D expenditures often follow a Rayleigh cumulative distribution function [1,2,6,7,10,11,12,20,21]. Watkins [21], Abernethy [1], Lee, Hogue and Hoffman [12], and Elrod [2] tested the ability of the Rayleigh model to fit actual weapon system R&D data. They all conclude that the Rayleigh model fit well. Lee, Hogue, and Gallagher [10, 11] presented a procedure, based on the Rayleigh model, to determine budget profiles from an R&D estimate.

The Rayleigh model for cumulative earned value during R&D is

$$v(t) = d[1 - \exp(-\alpha t^2)] \quad (1)$$

where  $v$  represents the earned value at time  $t$ . In this paper we model earned value by expenditures (as reported by ACWP) expressed in constant dollars. The parameter  $d$  scales the Rayleigh cumulative distribution function to costs, and the shape parameter,  $\alpha$ , determines the time of peak rate of expenditures,  $t_p$ :

$$\alpha = \frac{I}{2 t_p^2}. \quad (2)$$

Since the Rayleigh distribution function has an infinite tail, the modeled expenditures would never terminate. We define the time of final development,  $t_f$ , as when 97 percent of the expenditures are complete;

$$D = v(t_f) = 0.97d \quad (3)$$

where  $D$  is the total R&D program cost. The final time relates to the time of peak rate of expenditures with  $t_f = 2.65t_p$  [11]. In addition, the Rayleigh shape parameter  $\alpha$  can be determined from a projection of the completion time with

$$\alpha = \frac{3.5}{t_f^2}. \quad (4)$$

We employ the Rayleigh model to predict the change in earned value as time passes.

### 3. Earned Value Over Time

A generalized model that embraces both Rayleigh and Parr [19] models is

$$\frac{dv}{dt} = F(v) \quad (5)$$

where  $v$  is earned value and  $F(v)$  gives the rate at which the project absorbs resources efficiently.

The function  $F(v)$  is like Parr's "number of visible jobs" to which effort can efficiently be applied. The function  $F(v)$  must satisfy some common-sense conditions:  $F(v)$  must be positive, except that it is zero at  $v = v(t_f)$ , the final value of the project, and, possibly, also at  $v = v(t_0)$ , the project start.  $F(v)$  must be increasing in some neighborhood of  $v = v(t_0)$  and decreasing in some neighborhood of  $v = v(t_f)$ .

If  $F(v)$  is also continuous, (5) is uniquely soluble in the form

$$P(v) = t$$

where the continuously differentiable function  $P(v)$  satisfies  $dP/dF = 1/F$  with initial condition  $P(0) = 0$ . By the positivity of  $F$ ,  $P$  is monotone increasing, so the inverse function  $P^{-1}$  exists, and

$$v = P^{-1}(t).$$

This formulation is a generalization of the Rayleigh case shown in (1). A straightforward calculus exercise using (1) shows that the  $P(v)$  corresponding to Rayleigh is

$$P(v) = \left[ \frac{-1}{\alpha} \ln \left( 1 - \frac{v}{d} \right) \right]^{\frac{l}{2}},$$

and the  $F(v)$  for the Rayleigh case is

$$F(v) = 2\alpha d \left( 1 - \frac{v}{d} \right) \left[ \frac{-1}{\alpha} \ln \left( 1 - \frac{v}{d} \right) \right]^{\frac{l}{2}}.$$

Solving (5) with initial conditions of  $v(t_i) = v_i$  for the Rayleigh case, one gets

$$\begin{aligned} v(t) &= P^{-1}(t - t_i + P(v_i)) \\ &= d \left[ 1 - \exp \left( -\alpha \left( t - t_i + \sqrt{\frac{-1}{\alpha} \ln \left( 1 - \frac{v_i}{d} \right)} \right)^2 \right) \right] \end{aligned} \tag{6}$$

for  $t \geq t_i$ . We apply the Rayleigh model as employed in (6) to predict the earned value at a future time given an earlier estimate of the earned value. Equation (6) is the dynamics model that propagates state estimates (means of the Bayesian probability distribution functions) for earned value through time in the Kalman filter formulation.

#### 4. Kalman Filter

The Kalman filter is an iterative Bayesian state estimation technique. (Maybeck presents a thorough discussion in [15].) The state is the random variable of interest; in this application to R&D programs, the state is the earned value and the measurements are the reports of actual costs incurred. The first stage of the Kalman filter propagates the state distribution through time based on a dynamics model. The second stage updates the distribution with the information from an actual measurement of the system. The Kalman filter algorithm repeats these two steps for each available measurement. This section develops the propagation and update stages of a Kalman

filter. In this section, we assume the three parameters required in a Kalman filter exist; in the next section, we apply Multiple Model Adaptive Estimation (MMAE) [16,17], another Bayesian technique that uses many Kalman filters each with a different combination of assumed parameters, to evaluate the likelihood of various parameter values.

For this application, we define the Kalman filter state,  $x(t_i)$ , as the cumulative earned value (expenditures expressed in constant dollars) at time  $t_i$ ; thus,  $x(t_i) = v(t_i)$ . We indicate the means of Bayesian probability distributions for the Kalman filter state by a hat. At the time of each measurement, the Kalman filter algorithm calculates two state distribution means. A superscript minus sign indicates the distribution mean prior to incorporating the measurement update,  $\hat{x}(t_i^-)$ . Similarly, a superscript plus signs indicates the distribution mean updated with the information from a measurement at time  $t_i$ ,  $\hat{x}(t_i^+)$ .

The steps in a Kalman filter iterate between propagation of the distribution mean through time and measurement update of the distribution mean. The state propagation is determined for the Rayleigh model in (1) with (6) as

$$\hat{x}(t_i) = d \left[ 1 - \exp \left( -\alpha \left( t_i - t_{i-1} + \sqrt{\frac{-1}{\alpha} \ln \left( 1 - \frac{\hat{x}(t_{i-1}^+)}{d} \right)} \right)^2 \right) \right]. \quad (7)$$

The appropriate initial state distribution mean in this application is zero because no expenditures can be incurred before the beginning of the program;  $\hat{x}(t_0) = 0$ .

The measurement update step incorporates the new information from a measurement. The notation for the measurement at time  $t_i$  is  $z_i$ . In this application, the measurement is the value of ACWP reported in the cost performance reports, adjusted for inflation. Since the

measurement is a direct measure of the state, the Kalman filter residual is the difference between the measurement and the mean of the state distribution prior to incorporating the measurement:

$$r_i = z_i - \hat{x}(t_i^-). \quad (8)$$

The Kalman filter gain,  $k$ , weights the information provided by the dynamics model along with the prior measurements and the information provided by the new measurement. Thus, the Kalman filter algorithm calculates the updated state distribution mean with

$$\begin{aligned} \hat{x}(t_i^+) &= \hat{x}(t_i^-) + kr_i \\ &= (1 - k)\hat{x}(t_i^-) + k z_i. \end{aligned} \quad (9)$$

Since the Kalman filter gain provides the relative weighting of two pieces of information about the system available at the time, the gain is bounded between zero and one;  $0 \leq k \leq 1$ . If the gain is zero, the update distribution mean is based entirely on the dynamics model; whereas if the gain is one, the updated mean is the last measurement. With values for  $d$ ,  $\alpha$ , and  $k$ , one can apply a Kalman filter using (7), (8) and (9) iteratively for each available measurement (reported actual cost). The next section presents a development for Bayesian estimation of these three parameters.

## 5. Multiple Model Adaptive Estimation (MMAE)

MMAE is a Bayesian system identification technique that estimates unknown system parameters when applying Kalman filters [16,17]. In this application, we use MMAE to determine the likelihood for the parameters  $d$  (cost scale parameter),  $\alpha$  (Rayleigh shape parameter), and  $k$  (Kalman filter gain). The advantage of applying MMAE is that the

probabilities are conditional on the actual cost data, which prevents assigning probabilities to final costs below the incurred cost or completion times less than the elapsed duration.

An overview of the algorithm follows: The set-up for employing MMAE is to discretize the continuous space for each parameter into a set of representative points. The MMAE algorithm processes the measurements (reported actual costs in this application) through a Kalman filter at each combination of discrete parameters. Each filter's residuals determine the probability of that filter's parameters being correct, conditioned on the measurements processed to that time. After processing all the available measurements, the filter probabilities indicate the likelihood of the parameters in that filter being correct conditioned on the measurements. We relate the filter parameter  $d$  to total program cost with (3) and the filter parameter  $\alpha$  to project duration with the relationship in (4). We incrementally add the final filter probabilities as the filter parameters for  $d$  increase to generate curves depicting the cumulative probability of the final cost being less than any particular value. Similarly, we incrementally sum the filter probabilities as the values for  $\alpha$  increase to determine a likelihood curve for project duration.

The details of the algorithm begin with the set-up for applying MMAE, discretizing the parameter space. Define the number of Kalman filters as  $L$ . Let  $a_l$  represent the vector of parameters  $d_l$ ,  $\alpha_l$ , and  $k_l$  selected for the  $l$ th filter, where  $l = 1, \dots, L$ . With a vector of parameters  $a_l$ , one can process the data through a Kalman filter by iteratively applying (7), (8) and (9). In the examples and Monte Carlo analysis, we used 20 values for  $d$ , 20 values for  $\alpha$ , and 5 values for  $k$ , equally spaced in each dimension. Thus, we processed the reported cost data through 2,000 Kalman filters.

Our approach discretized the parameter space in two steps. The first step is processing the measurement data through filters with a coarse discretization, and the second step is refining the discretization based on the filters' sum of squared residuals. Let the measurement history be represented by  $Z_N = \{z_1, z_2, \dots, z_N\}$  where  $z_i$  is the cumulative cost incurred at time index  $t_i$ . We determine the range of the Rayleigh parameter  $\alpha$  from estimates of the minimum and maximum completion time with (4), and we varied the values of  $\alpha$  incrementally over the range. The default range for estimated completion times is from a minimum of the last cost report,  $t_N$ , to an arbitrary maximum time of 15 years. For example, if the maximum completion time is 15 years,  $\alpha_{\max} = \frac{3.5}{15^2} = 0.156$  from (4). ( $\alpha_{\max}$  is actually the smallest shape parameter.) Our algorithm sets the minimum value for the cost scale parameter equal to the last reported cost,  $d_1 = z_N$ , and sets the maximum value equal with the amount and time of the last cost report with the Rayleigh curve for the longest program,

$$d_m = \frac{z_N}{1 - \exp(-\alpha_{\max} t_N^2)}. \quad (10)$$

The Kalman filter gain ranges from 0 to 1. An analyst may adjust either the cost parameter or completion time ranges. The algorithm processes the cost data through each of the Kalman filters with this initial coarse discretization of the parameter space. We use this first pass through the data to estimate the residual variance and to refine the parameter discretization.

MMAE determines the filters' probabilities by the magnitude of that filter's residuals. The Kalman filter residuals for linear systems with known structural matrices and driven by white noise are independent and Gaussian distributed with zero mean and known variance [15].

Although these assumptions are not met in this case, other applications assumed that the residuals are Gaussian and obtained useful results [3,4,5,8,9]. We assumed that the residuals calculated with (8) are zero mean with a variance estimated from the Kalman filter with the smallest sum of squared residuals from an initial pass through the data;

$$s_r^2 = \min \left[ \frac{1}{N-L} \sum_{i=L}^N (r_i^L)^2 \mid L=1, \dots, L \right]. \quad (11)$$

After the first pass of the data through the bank of Kalman filters, we reduce the parameter range to eliminate parameter values that resulted in sum of squared residuals three times the minimum value,  $s_r^2$ . Our algorithm equally spaces the parameters for the Kalman filters across the reduced parameter ranges. The algorithm calculates the MMAE probabilities on the second pass of the data through the Kalman filters. Based on the assumption of zero mean and the residual variance estimated in (11), the Gaussian probability density function for the  $i$ th measurement,  $z_i$ , conditioned on the  $l$ th filter's vector of parameters,  $\mathbf{a}_l$ , and the prior measurement history,  $\mathbf{Z}_{i-1}$ , is

$$f(z_i \mid \mathbf{a}_l, \mathbf{Z}_{i-1}) = \frac{1}{\sqrt{2\pi s_r^2}} \exp\left(\frac{-r_i^2}{2 s_r^2}\right)$$

as adapted from Equation (10-98) in Reference [16]. The probability for the  $j$ th filter having the "correct" parameters conditioned on the measurement history through time  $t_i$  is

$$p_j(t_i \mid \mathbf{Z}_i) = \frac{f(z_i \mid \mathbf{a}_j, \mathbf{Z}_{i-1}) p_j(t_{i-1} \mid \mathbf{Z}_{i-1})}{\sum_{l=1}^L f(z_i \mid \mathbf{a}_l, \mathbf{Z}_{i-1}) p_l(t_{i-1} \mid \mathbf{Z}_{i-1})} \quad (12)$$

from Equation (10-104) in Reference [16]. The probabilities at each measurement time,  $t_i$  for  $i = 1, \dots, N$ , must sum to one;

$$\sum_{l=1}^L p_l(t_i) = 1. \quad (13)$$

The initial or *a priori* probabilities account for information available about the likelihood of particular filter combinations before the measurement data are processed. If no information is available, the *a priori* probabilities should all be equal;  $p_l(t_0) = 1/L$  for  $l = 1, \dots, L$ . In addition, if any of the filter probabilities became zero, that filter's probabilities, calculated with (12), would remain zero for all the later times. To prevent prematurely discarding potentially viable filter parameters, practitioners commonly apply a heuristic [16,17]; if any of the filter probabilities decreases below a very small lower bound, such as 0.0001, the heuristic artificially increases that filter's probability to the lower bound. The filter probabilities that result after the last datum are not adjusted with this heuristic. The final filter probabilities represent the likelihood of each combination of model parameters conditioned on the available measurement history,  $\mathbf{Z}_N$ .

We use the filter probabilities to determine estimates for the final cost and completion time. The final cost corresponding to the parameters  $d_l$  and  $\alpha_l$  is  $D_l = d_l T_{\alpha_l}$ ; the translation factor,  $T_{\alpha_l}$ , is 0.97 from (3) for  $D_l$  expressed in constant dollars. To express  $D_l$  in current dollars, the translation factor must account for inflation during the program. Let the sequence of start of fiscal years during the program duration be represented  $\tilde{t}_i$  with  $\tilde{t}_0$  being the program start and  $\tilde{t}_{\hat{N}}$  be the projected program end. Further, let the corresponding inflation indices for the following fiscal year be  $\tilde{r}_i$ . Then the translation factor corresponding to  $\alpha_l$  is  $T_{\alpha_l} = \sum_{i=0}^{\hat{N}-1} \tilde{r}_i (e^{-\alpha_l \tilde{r}_i^2} - e^{-\alpha_l \tilde{r}_{i+1}^2})$ . Each  $D_l$  should be constrained to be greater than or equal to last cost report, expressed appropriately in constant or current dollars. The mean estimate of final cost conditioned on the available measurement history,  $\mathbf{Z}_N$ , is calculated with

$$\hat{D} = \sum_{l=1}^L D_l p_l(t_N | Z_N) \quad (14)$$

where  $t_N$  is the time index corresponding to the last available cost report,  $z_N$ . Similarly, the conditional mean estimate for completion time, based on the (4), is

$$\hat{t}_f = \sum_{l=1}^L \left( \frac{3.5}{\alpha_l} \right)^{0.5} p_l(t_N | Z_N) \quad (15)$$

The program estimates from (14) and (15) are the most likely conditional on the actual cost data.

The cumulative probability cost curve conditioned on the measurement history shows the probability that the final cost,  $D$ , will be less than any dollar value. Let the cost scale parameters increase from  $d_1$  to  $d_m$ . The sum of filter probabilities for all the filters with  $d_i$  represents the probability over the range  $[0.5(d_{i-1} + d_i), 0.5(d_i + d_{i+1})]$  for  $i = 2, \dots, m-1$ . There is no conditional probability below  $d_1$  or above  $d_m$ . Define  $\tilde{d}_i = 0.5(d_i + d_{i+1})$  for  $i = 1, \dots, m-1$ , and  $\tilde{d}_m = d_m$ . The final cost estimate for the Rayleigh model with parameters  $d_l$  and  $\alpha_l$  is calculated as  $\tilde{D}_l = T_{\alpha_l} \tilde{d}_l$  where  $T_{\alpha_l}$  is the translator factor to constant or current dollars used in (14). The final cost estimate should exceed the last datum. We calculate the cumulative probabilities by summing the filter probabilities for filters with final cost estimates,  $\tilde{D}_l$ , less than a dummy cost variable  $\lambda$  with linear interpolation between cost estimates; with  $\tilde{D}_l$  for  $l = 1$  to  $L$  sorted in increasing magnitude:

$$P(D < \lambda | Z_N) = \begin{cases} 0 & \text{if } \lambda \leq \tilde{D}_1 \\ \frac{\lambda - \tilde{D}_1}{\tilde{D}_2 - \tilde{D}_1} p_1(t_N) & \text{if } \tilde{D}_1 \leq \lambda \leq \tilde{D}_2 \\ \sum_{j=1}^{l-1} p_j(t_N) + \frac{\lambda - \tilde{D}_l}{\tilde{D}_{l+1} - \tilde{D}_l} p_l(t_N) & \text{if } \tilde{D}_l \leq \lambda \leq \tilde{D}_{l+1} \text{ for } 2 \leq l \leq L-1 \\ 1 & \text{if } \lambda \geq \tilde{D}_L \end{cases} \quad (16)$$

The graph of  $P(D < \lambda | Z_N)$  versus  $\lambda$  shows the conditional probability that the final cost will be less than any particular value. Finer discretization smoothes the likelihood curve. The constant-dollar and current-dollar curves have very similar shapes. We also generate the cumulative probability curve for project duration using the parameter  $\alpha$  and the relationship in (4).

A confounding relationship limits the ability to estimate both  $d$  and  $\alpha$  when  $\alpha t^2$  is small [12]. This problem can be seen by expanding the exponential in (1);

$$v(t) = d(1 - \exp(-\alpha t^2)) = d \left[ 1 - \left( 1 - \alpha t^2 + \frac{\alpha^2 t^4}{2} \dots \right) \right] \approx \alpha d t^2 + O(d \alpha^2 t^4)$$

where the function  $O(\cdot)$  represents higher order terms. When  $\alpha t^2$  is small, the higher order terms are negligible and only the product of  $\alpha$  and  $d$ , but not their individual values, can be estimated from the data. The relationship  $\alpha t^2 < 0.5$  holds prior to the time of peak expenditure rate, as seen from (2). Thus, many different Rayleigh curves appear to fit the data from  $t_0$  to  $t_p$  due to the canceling effects of changes to  $\alpha$  and  $d$ . With an independent estimate for either the time of peak rate of expenditures or the completion time, an analyst may determine the parameter  $\alpha$  and estimate  $d$  using the data. MMAE has the same confounding problem as any statistical technique when only data before the peak expenditure rate is available. If an independent estimate of  $\alpha$  is

available, one can put that value into all the filters and apply MMAE to estimate the probability distribution of the final cost.

## 6. Algorithm Steps

While the development of the algorithm is complex, implementation is not difficult. An Excel spreadsheet with a Visual Basic Module that applies this technique is available from the authors. The runtime on a 486 computer is about 1 minute with 50 data points. The procedure steps are enumerated below:

### Step 1) Adjust the history of cost reports for inflation

- Determine the delta between cumulative cost reports
- Apply the appropriate inflation index to the delta
- Sum the constant dollar deltas to cumulative base-year costs
- Determine time indices in years for each datum from the program start date

### Step 2) Determine the completion time range (may be fixed to a single value)

- Default range is from the time index of the last cost report to 15 years (arbitrary)
- Adjust completion time range based on program knowledge
- Relate completion time range to corresponding  $\alpha$  range with (4)

### Step 3) Determine the range for final cost estimates

- Default for minimum value is last reported incurred costs (in constant dollars)
- Default for maximum value is estimated with (10)
- Adjust final cost range based on program knowledge
- Relate final cost range to range for  $d$  with (3)

**Step 4) Initialize Kalman filters**

- Set number of discrete points for each variable, such as 20 for  $d$  and  $\alpha$  with 5 for  $k$
- Determine discrete values equally spaced across selected parameter range
- Assign variables for a Kalman filter with each combination of parameter values
- Set prior mean of state distributions to zero at initial time index,  $t_0$ ;  $\hat{x}(t_0) = 0$

**Step 5) Process data through filters to estimate residual variance and adjust parameter ranges**

- Propagate state distribution means with (7)
- Update state distribution means with (8) and (9)
- Collect sum of squared residuals from (8) for each Kalman filter
- Find minimum sum of squared residuals and estimate residual variance,  $s_r^2$ , with (11)
- Reduce  $\alpha$  and  $d$  ranges to eliminate values that always resulted in sum of squared residuals greater than 3 times the minimum sum
- Equally space the filter parameters across the reduced parameter ranges
- Reset prior means and set filter probabilities  $p_l(t_0) = 1/L$  for  $l = 1, \dots, L$ .

**Step 6) Process data values through bank of filters to determine filter probabilities**

- Propagate state distribution means with (7)
- Update state distribution means with (8) and (9)
- Calculate filter probabilities with (12)
- Normalize filter probabilities to meet (13)
- Except for last data point, adjust filter probabilities for lower bound;  $p_l(t_i) \geq 0.0001$

**Step 7) Determine conditional probabilistic-weighted averages with (14) and (15)**

**Step 8) Determine conditional cost likelihood curve with (16)**

## 7. Sample Applications

We applied the Bayesian estimation approach to three diverse historic programs, the F-15 airframe development, the NavStar Global Positioning System (GPS) Satellite, and the MK 50 Torpedo. We selected these programs to cover a variety of technologies without prior knowledge of how well the Rayleigh model fit them. The F-15 development contract completed on schedule with very slight cost growth. The satellite program experienced much higher final cost than originally projected. The MK 50 program required a substantial schedule increase beyond the originally projected development time and almost twice the expense of original cost estimate. The only program data used was the originally projected duration and the actual cost reports. We set the completion time ranges from the originally projected length to twice that length in each application.

The F-15 airframe development contract started in January 1970. The contract continued

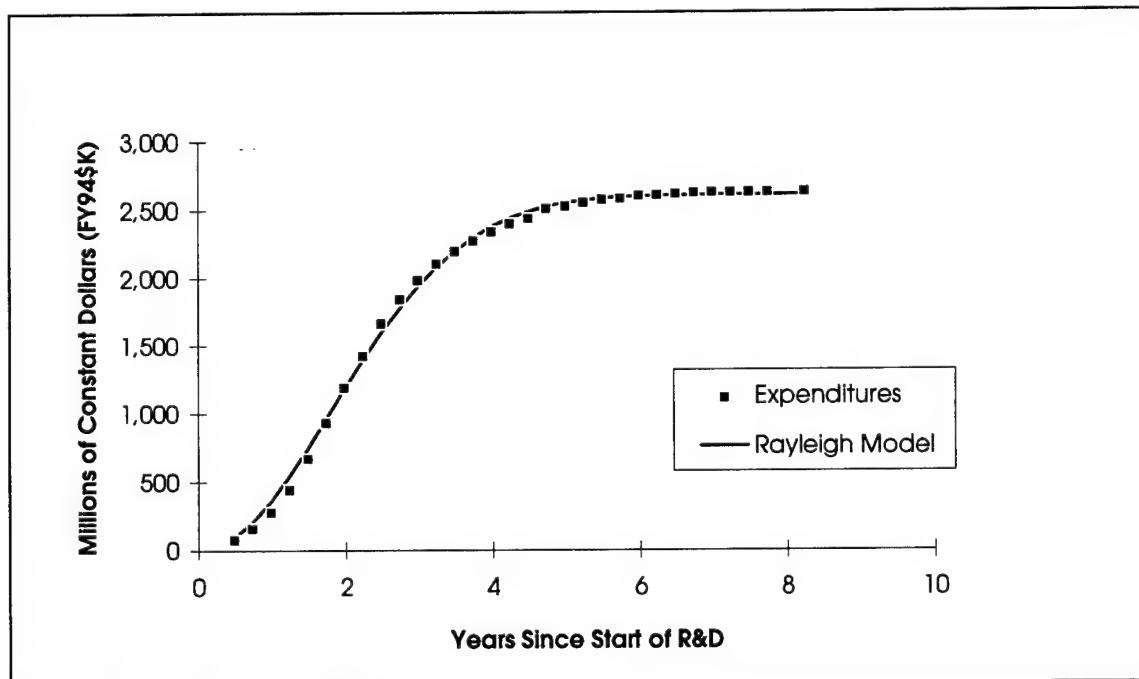


Figure 1. F-15 Airframe Contract Expenditures

for over 8 years, but most all the earned value occurred in the first 5 years. The Rayleigh model fits the reported expenditures reasonably. Figure 1 shows the Rayleigh model with the least squares parameters and the cost reports adjusted for inflation.

We applied our Bayesian cost estimation approach with the initial 3, 4, 5, and all years of F-15 airframe expenditures. We set the completion time range from 5 to 10 years. Figure 2 depicts the resulting likelihood curves. The likelihood curve based on only 3 years of data indicates a wide potential range for the final cost. When 4 or more years of data were used, the likelihood curves are very close to the actual final cost.

Most of the techniques used today to predict final costs of R&D programs give a point estimate for the final cost. Of course, the MMAE method gives much more than a point estimate. Nevertheless, to compare with other techniques, we had to select a point estimate. We compared the expected value from the current dollar likelihood curve with four techniques that

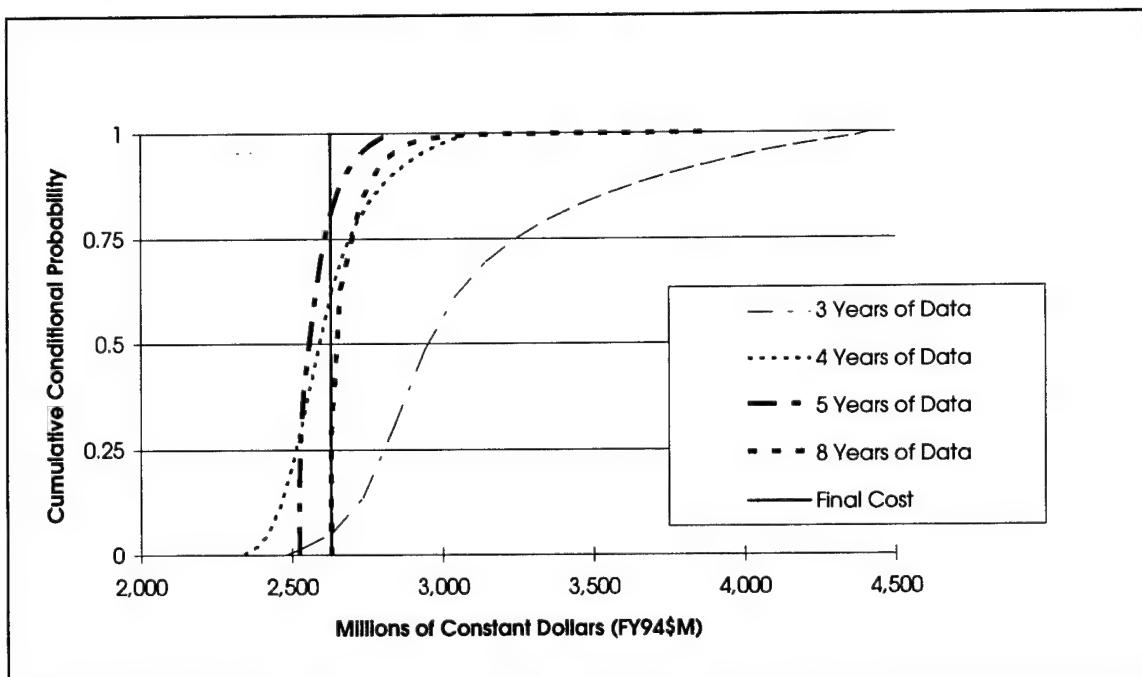


Figure 2. F-15 Airframe Development Final Cost Likelihood Curves

predict point estimates for final program cost.

For each of the four techniques, the final cost estimate is actual cost of work performed (ACWP) plus the quotient of work remaining divided by a cost performance index (CPI). Work remaining is budgeted work minus management reserve and budgeted cost of work performed (BCWP). The four techniques vary in the calculation of the CPI. The index for cumulative CPI (Cum CPI) is the cumulative BCWP performed divided by the cumulative ACWP. Similarly, CPI-3 and CPI-6 are calculated with BCWP and ACWP over the last 3 or 6 months, respectively. The cumulative CPI times cumulative schedule performance index (CPI\*SPI) is the CPI multiplied by the cumulative budget cost of work scheduled divided by the ACWP. We compare these four techniques with the expected value for the three historical programs.

Table 1 depicts the various final cost estimates. The CPI techniques were low with the initial cost data and increased over time. In contrast, the expected values from the Rayleigh and MMAE approach started much too high with only data prior to peak expenditure rate and decreased with additional data.

The second sample program is NavStar Global Positioning System (GPS) Satellite. This R&D program, which began in June of 1974, had a projected completion time of 4.3 years and a projected final cost of 40 million in then-year dollars. The program required almost 6 years and required 116.3 million in then-year dollars. The cumulative costs in constant dollars appear to fit

Table 1. F-15 Airframe Contract Final Cost Estimates (Current Dollars)

Years of Data	CUM CPI	CPI-3	CPI-6	CPI*SPI	Rayleigh/MMAE
2	779.9	752.4	764.8	784.7	1,880.4
3	775.9	779.4	775.3	777.3	1,016.3
4	678.4	696.8	689.4	681.2	834.6
5	815.6	820.9	819.0	816.2	836.3

The program manager estimate in Mar 1978 (8.25 years) was 850.0.

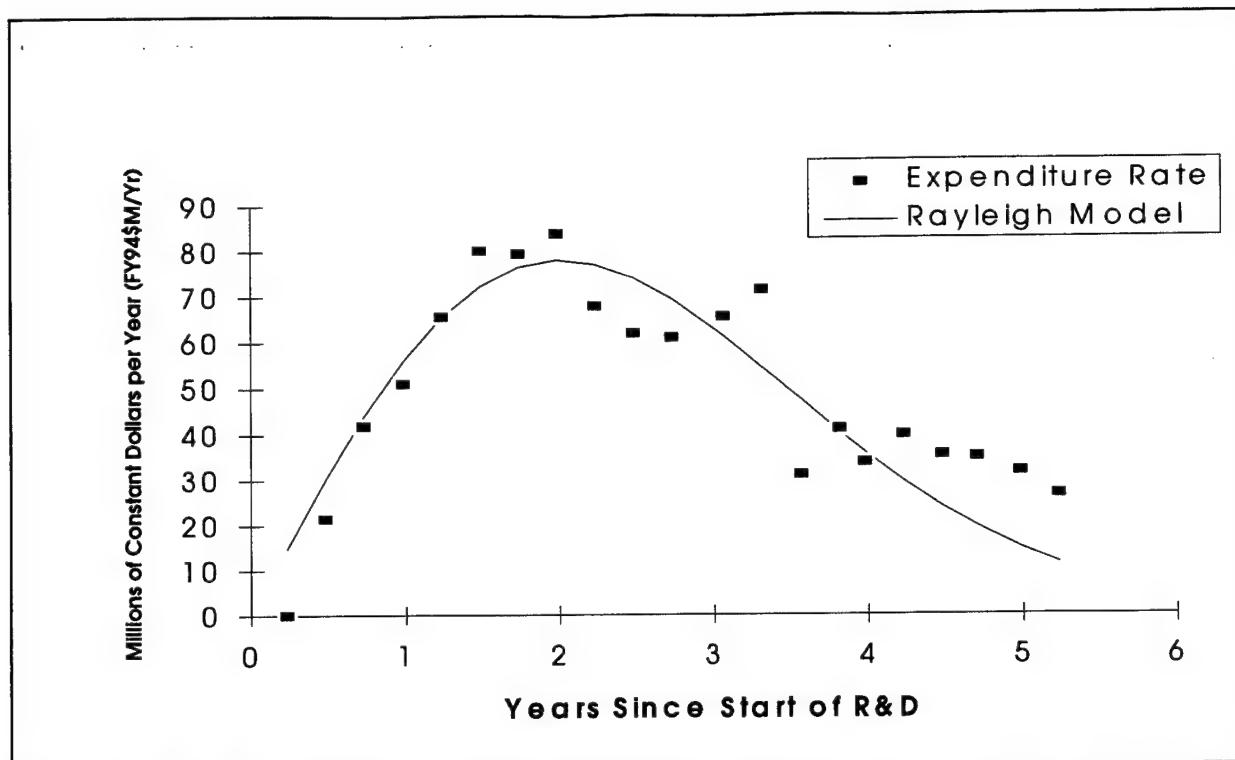


Figure 3. NavStar GPS Satellite Rate of Expenditures

the Rayleigh model; Figure 3 depicts the rate of expenditures and the derivative of the least squares Rayleigh model. We calculated the expenditure rates as the increase in reported cumulative expenditures divided by the time delta between cost reports. The Kalman filter gain,  $k$ , accounts for the measurement noise in the cost reports, apparent from the variation in reported expenditure rates. A quick heuristic, based on the Rayleigh model, to evaluate progress in R&D programs is that 60 percent of the expenditures occur after the time of peak expenditure rate,  $t_p$ .

We applied the Bayesian method with 2, 3, 4, and 5 years of expenditure data, and Figure 4 depicts the final cost likelihood curves. Without data after the peak rate of expenditure time, the completion time and final cost are statistically confounded. Since the peak rate occurs just after 2 years in the NavStar R&D, the likelihood cost curve based on 2 years of data indicates the potential for a very long and expensive program. The level expenditure rate during the fourth

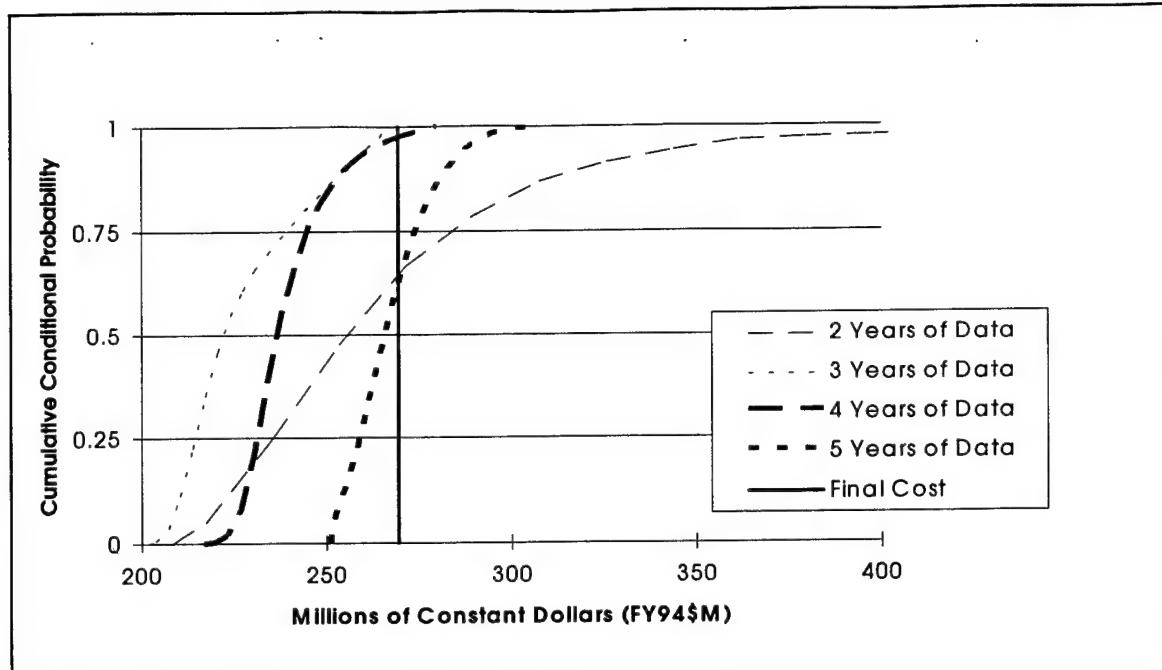


Figure 4. NavStar GPS Satellite Final Cost Likelihood Curves

year, shown in Figure 3, resulted in the likelihood curves based on 3 and 4 years of data to underestimate the final cost. With 5 years of data, the likelihood curve is very accurate.

We used the final filter probabilities from the same runs to generate the program duration likelihood curves. The duration range was from the original projection of 4.3 years to 8.6 years. The duration likelihood curves, shown in Figure 5, remain fairly consistent until 5 years of data was used. Figure 3 shows that the fourth year of data had a higher rate of expenditures than predicted with the Rayleigh model; the likelihood curves conditioned on 5 years of data indicate an increased probability of a longer program. Data fluctuations seems to affect the duration likelihood curves more than curves for final cost.

We present the various final cost estimates in Table 2. The CPI techniques were low initially and increased with more data. In contrast, the expected values from the proposed approach remained slightly below the actual final cost.

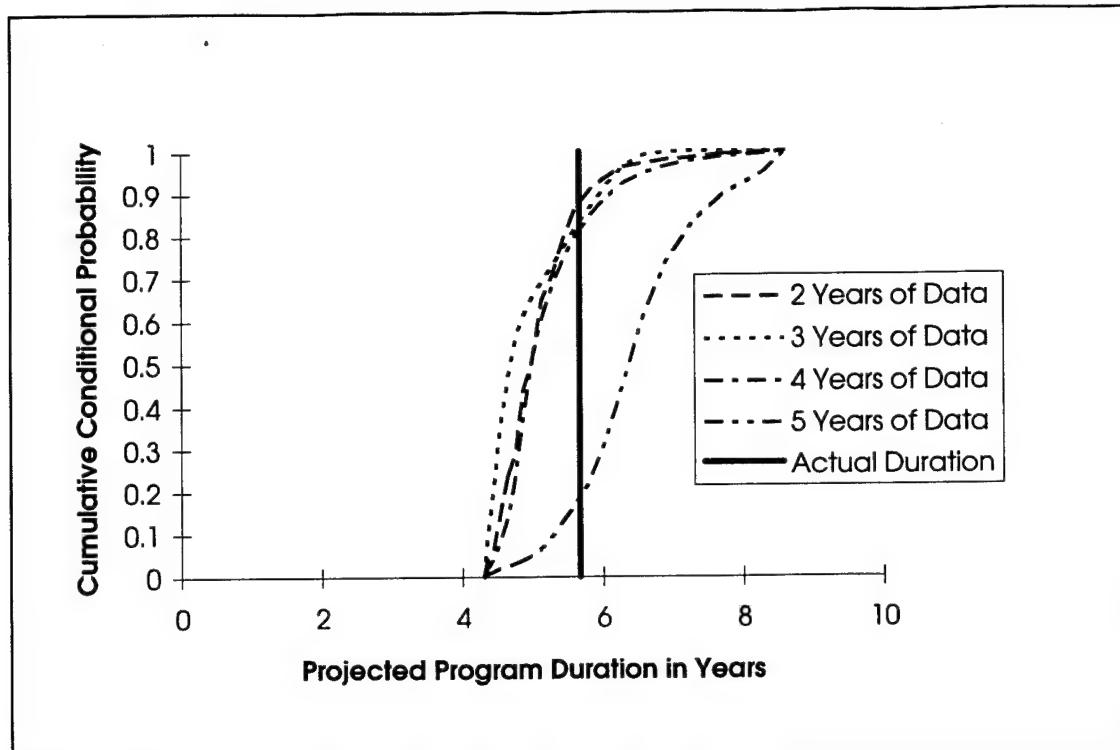


Figure 5. Navstar GPS Satellite Duration Likelihood Curves

The final example is the development of the MK 50 Torpedo. This program began in August of 1983 with a 5 year projected duration. The program was extended an additional 3 years, and the final costs increased 65 percent higher in current dollars. The completion time range was set from 5 to 10 years. The likelihood cost curves are depicted in Figure 6. A small probability of the cost being as high as the actual final cost is seen with even 3 years of data. With each year of additional data, the expected value from the likelihood cost curves moves closer to the actual final cost. With 6 and 7 years of data, much of the likelihood curves exceed

Table 2. NavStar GPS Satellite Final Cost Estimates

Years of Data	CUM CPI	CPI-3	CPI-6	CPI*SPI	Rayleigh/MMAE
2	70.0	80.9	78.7	71.0	109.8
3	99.8	100.3	101.7	103.3	96.2
4	104.4	108.7	104.3	106.9	98.6
5	114.0	114.5	114.4	115.5	112.2

The program manager estimate in Aug 1979 (5.25 years) was 116.3.

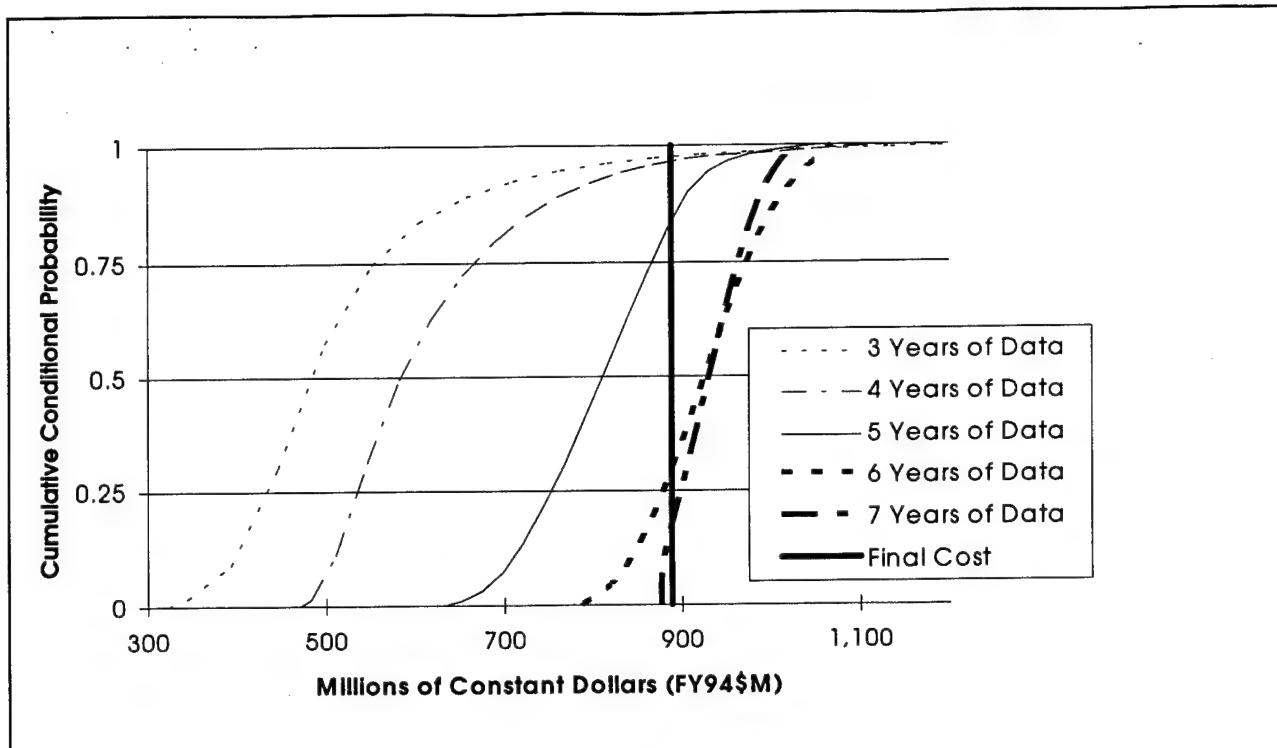


Figure 6. MK 50 Torpedo Final Cost Likelihood Curves

the final cost because the lower bound of the curves is cost incurred to that point in time.

The various final-cost point estimates, shown in Table 3, increased significantly. All the techniques started too low an increased as additional data was available.

These three examples demonstrate the capabilities of this Bayesian cost estimation approach for on-going R&D programs. In each of the applications, the algorithm made final cost likelihood curves that are very near the actual final costs based on very little program specific

Table 3. MK 50 Torpedo Final Cost Estimates

Years of Data	CUM CPI	CPI-3	CPI-6	CPI*SPI	Rayleigh/MMAE
3	580.7		566.3	589.0	409.8
4	529.8	540.1	536.8	527.8	559.8
5	655.7	667.1	659.4	650.6	629.4
6	685.3	678.8	680.8	685.0	746.3
7	707.3	706.9	706.6	708.9	714.5

The program manager estimate in Dec 1990 (7.25 years) was 711.4.

data. Tighter bounds on the possible range of final cost or completion times based on additional program knowledge would improve the proposed method's results. A Monte Carlo analysis, presented in the next section, shows the statistical effectiveness of this approach.

## 8. Monte Carlo Analysis

We conducted a Monte Carlo analysis of this technique with generated noise-corrupted Rayleigh data to verify its statistical validity. We evaluated the algorithm estimates for accuracy of point estimates and accuracy of the final cost likelihood curves. The performance statistics were collected after applying the algorithm with various amounts of the generated data.

Various final costs, completion times, and noise levels determined specific cases. We generated quarterly data such that the initial cost at time zero was zero, the cumulative cost always increased, and the cost at completion time was the final cost. For each cost report, the generated datum was calculated with

$$z_i = v(t_i) = d[F(t_{i-1}) + (F(t_i) - F(t_{i-1}))(1 + \epsilon)]$$

such that  $v(t_0) = 0$ ,  $v(t_f) = D$ , and  $v(t_i) \geq v(t_{i-1})$

where  $F(t) = 1 - \exp(-\alpha t^2)$ , the cumulative Rayleigh distribution function,  $d$  is from (3),  $\alpha$  is from (4), and  $\epsilon$  is a uniform random variable between plus and minus the noise level.

We tested seven cases. The final costs used were 2,000, 1,500 and 1,000 for a 12 year program. The Rayleigh shape parameters were determined with (4), and the noise level for 5 cases was set at 0.1. We varied the noise level to 0.2 and 0.3 for the 12 year program with final cost of 1,000 dollars. We also varied the completion time for the 1,000 dollar program to 9 and

to 6 years. For each case, summary statistics were collected across 500 data sets; we applied the algorithm both with and without using the known completion times. In all the tables to be presented, the first three columns define the case by giving the true final cost, true program completion time and the noise level used to generate the data. The next sets of columns show results based on increasing amounts of data used in the estimates. For example, the column with “Time of Estimate” of 3 indicates that 3 years of quarterly data were used to calculate the statistics in that column. We define errors as the estimated value minus the true value. The top halves of the tables are results based on estimated completion times, and the bottom halves present the results when the program completion time is known, in essence estimated perfectly.

The first measure of effectiveness is the accuracy of the probabilistic mean in estimating the true cost used to generate the data. We calculated the probabilistic mean with (15) and adjusted to final cost with (3). Table 4 shows the statistics for the seven cases. For a 12 year program with unknown completion time, the results with 3 years of data have large errors and corresponding large standard deviations. This is a result of the statistical indeterminacy between the cost scale parameter and the Rayleigh shape parameter. If the final time of the program is known, the errors in the final costs are very small as seen in the bottom half of Table 4. The errors with unknown completion times are conservative in that they estimate the program to be much higher in cost and longer than it actually was. With data that encompass half the actual completion time, the errors become very small in comparison with the final cost with relatively small variance.

The first three cases show the linear effect for changes in the true final cost. Since we used the same seed in the random number generator, the error statistics are exactly proportional

Table 4. Probabilistic Mean Estimator Statistics

Case			Average Errors				Error Standard Deviations			
Final Cost	Final Time	Noise Level	Time of Estimate				Time of Estimate			
			3	6	9	12	3	6	9	12
Estimated Final Cost and Estimated Final Time										
2,000	12	0.1	434.0	5.7	1.3	0.3	562.6	13.1	2.0	0.3
1,500	12	0.1	325.6	4.3	0.9	0.2	422.1	9.8	1.5	0.2
1,000	12	0.1	217.1	2.9	0.6	0.1	281.4	6.6	1.0	0.2
1,000	12	0.2	208.9	3.2	0.0	0.7	302.6	18.9	3.0	0.6
1,000	12	0.3	161.9	8.2	0.1	1.0	312.7	44.2	5.3	0.9
1,000	9	0.1	70.1	3.9	0.2		186.6	9.9	0.3	
1,000	6	0.1	11.8	1.8			31.8	0.5		
Estimated Final Cost with Given Final Time										
2,000	12	0.1	-0.3	-0.1	0.0	0.1	20.7	4.5	1.2	0.2
1,500	12	0.1	-0.2	-0.1	0.0	0.1	15.5	3.4	0.9	0.2
1,000	12	0.1	-0.2	-0.1	-0.1	0.0	10.4	2.3	0.6	0.1
1,000	12	0.2	0.4	0.0	0.2	0.4	20.3	5.0	1.4	0.4
1,000	12	0.3	3.3	-0.1	-0.1	0.5	31.0	7.6	1.9	0.5
1,000	9	0.1	-1.1	0.2	0.1		9.0	2.2	0.2	
1,000	6	0.1	-8.2	0.3			6.1	0.3		

to the true final cost. The third through fifth cases show that as the noise levels increase so do the estimate standard deviations. The errors for the shorter programs in the last two cases are less because proportionately more program data was used for the estimates. In all cases, the error statistics improve with additional data, and the errors are very small when the completion time was known. We did not include the results for the median because of their similarity.

The second statistics depict the effectiveness in quantifying the cost risk of continuing the program. The cost risk is depicted with the cumulative cost curves generated with (16). We evaluated the curves by collecting the frequency with which the true cost was less than the predicted 30th and 70th percentiles. Table 5 shows the statistics for 500 runs for each case. When the reported frequency for the 30th percentile exceeds 0.30, the curve estimates were too high. Following the trend of the mean and median, the 30th percentile was high initially and

Table 5. Estimated Percentile Efficiencies

Case			Frequency < 30th Percentile				Frequency $\leq$ 70th Percentile			
Final Cost	Final Time	Noise Level	Time of Estimate				Time of Estimate			
			3	6	9	12	3	6	9	12
Estimated Final Cost and Estimated Final Time										
2,000	12	0.1	0.54	0.38	0.26	1.00	0.75	0.96	0.98	1.00
1,500	12	0.1	0.54	0.38	0.26	1.00	0.75	0.96	0.98	1.00
1,000	12	0.1	0.54	0.38	0.26	1.00	0.75	0.96	0.98	1.00
1,000	12	0.2	0.57	0.31	0.10	1.00	0.79	0.95	0.83	1.00
1,000	12	0.3	0.50	0.37	0.13	1.00	0.75	0.82	0.85	1.00
1,000	9	0.1	0.26	0.48	1.00		0.71	0.87	1.00	
1,000	6	0.1	0.47	1.00			0.90	1.00		
Estimated Final Cost with Given Final Time										
2,000	12	0.1	0.23	0.17	0.11	1.00	0.77	0.82	0.92	1.00
1,500	12	0.1	0.23	0.17	0.11	1.00	0.77	0.82	0.92	1.00
1,000	12	0.1	0.23	0.17	0.11	1.00	0.77	0.82	0.92	1.00
1,000	12	0.2	0.29	0.22	0.17	1.00	0.69	0.79	0.85	1.00
1,000	12	0.3	0.32	0.21	0.10	1.00	0.72	0.78	0.86	1.00
1,000	9	0.1	0.43	0.15	0.98		0.47	0.88	1.00	
1,000	6	0.1	0.10	0.99			0.10	1.00		

Note: The theoretical standard deviation of these frequencies is 0.0205.

decreased as the amount of data increased. When all the data was used, the entire cumulative cost curve exceeds the value of the last data point, which was the true final cost, because the cumulative cost projects always exceed reported incurred costs. When the final time was known, the 30th percentiles were slightly low and the 70th percentiles were slightly high.

The final measure of effectiveness is the width of the 40 percent confidence interval that could be formed from the 30th to the 70th percentiles. The confidence interval widths indicate the accuracy the algorithm assigns to mean estimates in Table 4. Table 5 shows that these assigned accuracies are commensurate with their true accuracies. Table 6 shows that as the additional data was used in the algorithm the confidence interval widths become very small. Using all the data, the point estimator error was less than 0.2 percent of the true final cost, and the corresponding 40 percent confidence was less than 0.4 percent of the true final cost.

Table 6, Estimated 40 Percent Confidence Interval Width

Case			Confidence Interval Width (Distance Between 30th and 70th Percentiles)			
Final Cost	Final Time	Noise Level	3	6	9	12
Estimated Final Cost and Estimated Final Time						
2,000	12	0.1	343.5	22.9	7.1	1.0
1,500	12	0.1	257.4	17.2	5.4	0.8
1,000	12	0.1	171.6	11.5	3.6	0.5
1,000	12	0.2	197.7	20.9	7.8	1.2
1,000	12	0.3	226.4	48.2	10.9	1.6
1,000	9	0.1	197.3	16.5	1.0	
1,000	6	0.1	33.9	3.4		
Estimated Final Cost with Given Final Time						
2,000	12	0.1	25.1	7.7	4.0	0.7
1,500	12	0.1	18.8	5.8	3.0	0.8
1,000	12	0.1	12.6	3.8	2.0	0.4
1,000	12	0.2	20.7	7.6	3.5	1.0
1,000	12	0.3	29.1	11.3	5.4	1.2
1,000	9	0.1	1.5	2.8	1.0	
1,000	6	0.1	0.0	2.5		

## 9. Summary

We developed and tested a method of estimating the probability of final cost and completion time for R&D programs conditioned on actual cost reports. The method is based on assuming that the cumulative earned value (represented by constant-dollar expenditures) of the development program followed a Rayleigh distribution. The approach uses Multiple Model Adaptive Estimation (MMAE), which employs a large number of Kalman filters, to estimate the Rayleigh model parameters. The MMAE technique, as applied in this application, provides the probabilities of various final cost estimates and projected completion times conditioned on actual cost data. We summed those probabilities to produce final cost likelihood curves. These curves depict the likelihood the final cost estimate will be below various cost estimates. The final cost

estimates and likelihood curves can be converted from constant dollars to current dollars.

Similarly, likelihood curves for completion time can be constructed.

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